The Radicals of Crossed Products *

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Abstract

The relations between the radical of crossed product $R\#_\sigma H$ and algebra R are obtained. Using this theory, the author shows that if H is a finite-dimensional semisimple, cosemisimle, and either commutative or cocommutative Hopf algebra, then R is H-semiprime iff R is semiprime iff $R\#_\sigma H$ is semiprime.

0 Introduction and Preliminaries

J.R. Fisher [7] built up the general theory of H-radicals for H-module algebras. He studied H-Jacobson radical and obtained

$$r_j(R\#H) \cap R = r_{Hj}(R) \tag{1}$$

for any irreducible Hopf algebra H([7, Theorem 4]). J.R. Fisher [7] asked when is

$$r_j(R\#H) = r_{Hj}(R)\#H \tag{2}$$

and asked if

$$r_j(R\#H) \subseteq (r_j(R) : H)\#H \tag{3}$$

R.J. Blattner, M. Cohen and S. Montgomery in [3] asked whether $R\#_\sigma H$ is semiprime with a finite-dimensional semisimple Hopf algebra H when R is semiprime, which is called the semiprime problem.

If H is a finite-dimensional semisimple Hopf algebra and R is semiprime, then $R\#_\sigma H$ is semiprime in the following five cases:

(i) k is a perfect field and H is cocommutative;

(ii) H is irreducible cocommutative;

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(iii) The weak action of H on R is inner;

(iv) $H = (kG)^*$, where G is a finite group;

(v) H is cocommutative.

Part (i) (ii) are due to W. Chin [4, Theorem 2, Corollary 1]. Part (iii) is due to B.J. Blattner and S. Montgomery [2, Theorem 2.7]. Part (iv) is due to M. Cohen and S. Montgomery [6, Theorem 2.9]. Part (v) is due to S. Montgomery and H.J. Schneider [9, Corollary 7.13].

If $H = (kG)^*$, then relation (2) holds, due to M. Cohen and S. Montgomery [6, Theorem 4.1]

In this paper, we obtain the relation between H-radical of H-module algebra R and radical of $R\#H$. We give some sufficient conditions for (2) and (3) and the formulae, which are similar to (1), (2) and (3) for H-prime radical respectively. We show that (1) holds for any Hopf algebra H. Using radical theory and the conclusions in [9], we also obtain that if H is a finite-dimensional semisimple, cosemisimle and either commutative or cocommutative Hopf algebra, then R is H-semiprime iff R is semiprime iff $R\#_\sigma H$ is semiprime.

In this paper, unless otherwise stated, let k be a field, R be an algebra with unit over k, H be a Hopf algebra over k and H^* denote the dual space of H.

R is called a twisted H-module algebra if the following conditions are satisfied:

(i) H weakly acts on R;

(ii) R is a twisted H-module, that is, there exists a linear map $\sigma \in Hom_k(H \otimes H, R)$ such that $h \cdot (k \cdot r) = \sum \sigma(h_1, k_1)(h_2k_2 \cdot r)\sigma^{-1}(h_3, k_3)$ for all $h, k \in H$ and $r \in R$.

It is clear that if σ is trivial, then twisted H-module algebra R is an H-module algebra.

Set

$$Spec(R) = \{I \mid I \text{ is a prime ideal of } R\};$$

$$H\text{-}Spec(R) = \{I \mid I \text{ is an } H\text{-prime ideal of } R\}.$$

1 The Baer radical of twisted H-module algebras

In this section, let k be a commutative associative ring with unit, H be an algebra with unit and comultiplication $\triangle$, R be an algebra over k (R may be without unit) and R be a twisted H-module algebra.

Definition 1.1 $r_{Hb}(R) := \cap\{I \mid I$ *is an* H*-semiprime ideal of* R *};*

$r_{bH}(R) := (r_b(R) : H)$

$r_{Hb}(R)$ *is called the* H*-Baer radical (or* H*-prime radical) of twisted* H*-module algebra* R*.*

Lemma 1.2 *(1) If E is a non-empty subset of R, then $(E) = (H \cdot E) + R(H \cdot E) + (H \cdot E)R + R(H \cdot E)R$, where (E) denotes the H-ideal generated by E in R;*

(2) If I is a nilpotent H-ideal of R, then $I \subseteq r_{Hb}(R)$.

Proof. (1) It is trivial.

(2) If I is a nilpotent H-ideal and P is an H-semiprime ideal, then $(I + P)/P$ is nilpotent simply because $(I + P)/P \cong I/(I \cap P)$ (as algebras) . Thus $I \subseteq P$ and $I \subseteq r_{Hb}(R)$. □

Proposition 1.3 *(1) $r_{Hb}(R) = 0$ iff R is H-semiprime;*

(2) $r_{Hb}(R/r_{Hb}(R)) = 0$;

(3) R is H-semiprime iff $(H \cdot a)R(H \cdot a) = 0$ always implies $a = 0$ for any $a \in R$; R is H-prime iff $(H \cdot a)R(H \cdot b) = 0$ always implies $a = 0$ or $b = 0$ for any a, $b \in R$;

(4) If R is H-semiprime, then $W_H(R) = 0$.

Proof. (1) If $r_{Hb}(R) = 0$, then R is H-semiprime by Lemma 1.2 (2). Conversely, if R is H-semiprime, then 0 is an H-semiprime ideal and so $r_{Hb}(R) = 0$ by Definition 1.1.

(2) If $B/r_{Hb}(R)$ is a nilpotent H-ideal of $R/r_{Hb}(R)$, then $B^k \subseteq r_{Hb}(R)$ for some natural number k and so $B \subseteq r_{Hb}(R)$, which implies that $R/r_{Hb}(R)$ is H-semiprime. Thus $r_{Hb}(R/r_{Hb}(R)) = 0$ by part (1).

(3) If R is H-prime and $(H \cdot a)R(H \cdot b) = 0$ for a and $b \in R$, then $(a)^2(b)^2 = 0$ by Lemma 1.2 (1), where (a) and (b) are the H-ideals generated by a and b in R respectively. Since R is H-prime, $a = 0$ or $b = 0$. Conversely, if both B and C are H-ideals of R and $BC = 0$, then $(H \cdot a)R(H \cdot b) = 0$ and $a = 0$ or $b = 0$ for any $a \in B$ and $b \in C$, which implies that $B = 0$ or $C = 0$. Thus R is an H-prime. Similarly, the other assertion holds.

(4) For any $0 \neq a \in R$, there exist $b_1 \in R$ and $h_1, h_1' \in H$ such that $0 \neq a_2 = (h_1 \cdot a_1)b_1(h_1' \cdot a_1) \in (H \cdot a_1)R(H \cdot a_1)$ by part (3), where $a_1 = a$. Similarly, for $0 \neq a_2 \in R$, there exist $b_2 \in A$, h_2, $h_2' \in H$ such that $0 \neq a_3 = (h_2 \cdot a_2)b_2(h_2' \cdot a_2) \in (H \cdot a_2)R(H \cdot a_2)$, which implies that there exists an H-m-sequence $\{a_n\}$ such that $a_n \neq 0$ for any natural number n. Thus $W_H(R) = 0$. □

Theorem 1.4 *$r_{Hb}(R) = W_H(R) = \cap\{I \mid I$ is an H-prime ideal of $R\}$.*

Proof. Let $D = \cap\{I \mid I$ is an H-prime ideal of R $\}$. Obviously, $r_{Hb}(R) \subseteq D$.

If $0 \neq a \notin W_H(R)$, then there exists an m-sequence $\{a_i\}$ in R with $a_1 = a$ and $a_{n+1} = (h_n \cdot a_n)b_n(h_n' \cdot a_n) \neq 0$ for $n = 1, 2, \cdots$. Let $\mathcal{F} = \{I \mid I$ is an H-ideal of R and $I \cap \{a_1, a_2, \cdots\} = \emptyset\}$. By Zorn's Lemma, there exists a maximal P in $\mathcal{F}$. If both I and J are H-ideals of R with $I \not\subseteq P$ and $J \not\subseteq P$ such that $IJ \subseteq P$, then there exist natural numbers n and m such that $a_n \in I + P$ and $a_m \in J + P$. Since $a_{n+m+1} = (h_{n+m} \cdot a_{n+m})b_{n+m}(h_{n+m}' \cdot a_{n+m}) \in (I + P)(J + P) \subseteq P$, we get a contradiction.

Thus P is an H-prime ideal of R. Obviously, $a \notin P$, which implies that $a \notin D$. Therefore $D \subseteq W_H(R)$.

For any $x \in W_H(R)$, let $\bar{R} = R/r_{Hb}(R)$. It follows from Proposition 1.3 (1) (2) (4) that $W_H(\bar{R}) = 0$. For any H-m-sequence $\{\bar{a}_n\}$ with $\bar{a}_1 = \bar{x}$ in $\bar{R}$, there exist $\overline{b}_n \in \overline{R}$ and $h_n, h'_n \in H$ such that $\overline{a}_{n+1} = (h_n \cdot \overline{a}_n)\overline{b}_n(h'_n \cdot \overline{a}_n)$ for any natural number n. Let $a'_1 = x$ and $a'_{n+1} = (h_n \cdot a'_n)b_n(h'_n \cdot a'_n)$ for any natural number n. Since $\{a'_n\}$ is an H-m-sequence with $a'_1 = x$ in R, there exists a natural number k such that $a'_k = 0$. It is clear that $\overline{a}_n = \overline{a'_n}$ for any natural number n by induction. Thus $\bar{a}_k = 0$ and $\overline{x} \in W_H(\overline{R})$. Considering $W_H(\overline{R}) = 0$, we have $x \in r_{Hb}(R)$, which implies that $W_H(R) \subseteq r_{Hb}(R)$. Therefore $W_H(R) = r_{Hb}(R) = D$. $\Box$

2 The Baer and Jacobson radicals of crossed products

By [10, Lemma 7.1.2], if $R\#_\sigma H$ is crossed product defined in [10, Definition 7.1.1], then R is a twisted H-module algebra.

Let R be an algebra and $M_{m\times n}(R)$ be the algebra of $m \times n$ matrices with entries in R. For $i = 1, 2, \cdots m$ and $j = 1, 2, \cdots n$. Let $(e_{ij})_{m\times n}$ denote the matrix in $M_{m\times n}(R)$, where (i, j)-entry is 1_R and the others are zero. Set

$\mathcal{I}(R) = \{I \mid I \text{ is an ideal of } R\}$.

$r_{Hj}(R) := r_j(R\#_\sigma H) \cap R$;

$r_{jH}(R) := (r_j(R) : H)$.

Lemma 2.1 *Let M_R be a free R-module with finite rank and $R' = End(M_R)$. Then there exists a unique bijective map*

$$\Phi : \mathcal{I}(R) \longrightarrow \mathcal{I}(R')$$

such that $\Phi(I)M = MI$ *and*

(1) Φ is a map preserving containments, finite products, and infinite intersections;

(2) $\Phi(I) \cong M_{n\times n}(I)$ for any ideal I of R;

(3) I is a (semi)prime ideal of R iff $\Phi(I)$ is a (semi)prime ideal of R';

(4) $r_b(R') = \Phi(r_b(R))$;

(5) $r_j(R') = \Phi(r_j(R))$.

Proof Since M_R is a free R-module with rank n, we can assume $M = M_{n\times 1}(R)$. Thus $R' = End(M_R) = M_{n\times n}(R)$ and the module operation of M over R becomes the matrix operation. Set $M' = M_{1\times n}(R)$. Obviously, $M'M = R$. Since $(e_{i1})_{n\times 1}(e_{1j})_{1\times n} = (e_{ij})_{n\times n}$ for $i, j = 1, 2, \cdots n$, $MM' = M_{n\times n}(R) = R'$. Define

$$\Phi(I) = MIM'$$

for any ideal I of R. By simple computation, we have that $\Phi(I)$ is an ideal of R' and $\Phi(I)M = MI$. If J is an ideal of R' such that $JM = MI$, then $JM = \Phi(I)M$ and $J = \Phi(I)$, which implies Φ is unique. In order to show that Φ is a bijection from $\mathcal{I}(R)$ onto $\mathcal{I}(R')$, we define a map Ψ from $\mathcal{I}(R')$ to $\mathcal{I}(R)$ sending I' to $M'I'M$ for any ideal I' of R'. Since $\Phi\Psi(I') = MM'I'MM' = I'$ and $\Psi\Phi(I) = M'MIM'M = I$ for any ideal I' of R' and ideal I of R, we have that Φ is bijective.

(1) Obviously Φ preserves containments. We see that

$$\Phi(IJ) = MIJM' = (MIM')(MJM') = \Phi(I)\Phi(J)$$

for any ideals I and J of R. Thus Φ preserves finite products. To show that Φ preserves infinite intersections, we first show that

$$M(\cap\{I_\alpha \mid \alpha \in \Omega\}) = \cap\{MI_\alpha \mid \alpha \in \Omega\} \tag{4}$$

for any $\{I_\alpha \mid \alpha \in \Omega\} \subseteq \mathcal{I}(R)$. Obviously, the right side of relation (4) contains the left side of relation (4). Let $\{u_1, u_2, \cdots, u_n\}$ be a basis of M over R. For any $x \in \cap\{MI_\alpha \mid \alpha \in \Omega\}$, any $\alpha, \alpha' \in \Omega$, there exist $r_i \in I_\alpha$ and $r_i' \in I_{\alpha'}$ such that $x = \sum u_i r_i = \sum u_i r_i'$. Since $\{u_i\}$ is a basis, $r_i = r_i'$, which implies $x \in M(\cap\{I_\alpha \mid \alpha \in \Omega\})$. Thus the relation (4) holds. It follows from relation (4) that

$$\Phi(\cap\{I_\alpha \mid \alpha \in \Omega\})M = \cap\{\Phi(I_\alpha)M \mid \alpha \in \Omega\}$$

Since $\Phi(\cap\{I_\alpha \mid \alpha \in \Omega\})M = \cap\{\Phi(I_\alpha)M \mid \alpha \in \Omega\} \supseteq (\cap\{\Phi(I_\alpha) \mid \alpha \in \Omega\})M$, we have that

$$\Phi(\cap\{I_\alpha \mid \alpha \in \Omega\}) \supseteq \cap\{\Phi(I_\alpha) \mid \alpha \in \Omega\}.$$

Obviously,

$$\Phi(\cap\{I_\alpha \mid \alpha \in \Omega\}) \subseteq \cap\{\Phi(I_\alpha) \mid \alpha \in \Omega\}.$$

Thus

$$\Phi(\cap\{I_\alpha \mid \alpha \in \Omega\}) = \cap\{\Phi(I_\alpha) \mid \alpha \in \Omega\}.$$

(2) Obviously, $\Phi(I) = MIM' = M_{n\times 1}(R)IM_{1\times n}(R) \subseteq M_{n\times n}(I)$. Since $a(e_{ij})_{n\times n} = (e_{i1})_{n\times 1}a(e_{1j})_{1\times n} \in MIM'$ for all $a \in I$ and $i, j = 1, 2, \cdots n$,

$$\Phi(I) = MIM' = M_{n\times 1}(R)IM_{1\times n}(R) \supseteq M_{n\times n}(I).$$

Thus part (2) holds.

(3) Since bijection Φ preserves products, part (3) holds.

(4) We see that

$$\begin{aligned}
\Phi(r_b(R)) &= \Phi(\cap\{I \mid I \text{ is a prime ideal of } R\}) \\
&= \cap\{\Phi(I) \mid I \text{ is a prime ideal of } R\} \text{ by part (1)} \\
&= \cap\{\Phi(I) \mid \Phi(I) \text{ is a prime ideal of } R'\} \text{ by part (3)} \\
&= \cap\{I' \mid I' \text{ is a prime ideal of } R'\} \text{ since } \Phi \text{ is surjective} \\
&= r_b(R')
\end{aligned}$$

(5) We see that

$$\begin{aligned}\Phi(r_j(R)) &= M_{n\times n}(r_j(R)) \text{ by part (2)}\\ &= r_j(M_{n\times n}(R)) \text{ by [15, Theorem 30.1]}\\ &= r_j(R')\ . \qquad \Box\end{aligned}$$

Let H be a finite-dimensional Hopf algebra and $A = R\#_\sigma H$. Then A is a free right R-module with finite rank by [10, Proposition 7.2.11] and $End(A_R) \cong (R\#_\sigma H)\#H^*$ by [10, Corollary 9.4.17]. By part (a) in the proof of [9, Theorem 7.2], it follows that Φ in Lemma 1.2 is the same as in [9, Theorem 7.2].

Lemma 2.2 *Let H be a finite-dimensional Hopf algebra and $A = R\#_\sigma H$. Then*
(1) If P is an H^*-ideal of A, then $P = (P \cap R)\#_\sigma H$.
(2) $\Phi(I) = (I\#_\sigma H)\#H^*$ for every H-ideal I of R;
(3)

$$\{P \mid P \textit{ is an } H\textit{-ideal of } A\#H^*\} = \{(I\#_\sigma H)\#H^* \mid I \textit{ is an } H\textit{-ideal of } R\} \tag{5}$$

$$\{P \mid P \textit{ is an } H^*\textit{-ideal of } A\} = \{I\#_\sigma H \mid I \textit{ is an } H\textit{-ideal of } R\} \tag{6}$$

$$\begin{aligned}&\{P \mid P \textit{ is an } H\textit{-prime ideal of } A\#H^*\}\\ &= \{(I\#_\sigma H)\#H^* \mid I \textit{ is an } H\textit{-prime ideal of } R\}\end{aligned} \tag{7}$$

(4) H-Spec$(R) = \{(I : H) \mid I \in$ Spec $(R)\}$;
(5)

$$(\cap\{I_\alpha \mid \alpha \in \Omega\} : H) = \cap\{(I_\alpha : H) \mid \alpha \in \Omega\} \tag{8}$$

where I_α is an ideal of R for all $\alpha \in \Omega$;
(6)

$$(\cap\{I_\alpha \mid \alpha \in \Omega\})\#_\alpha H = \cap\{(I_\alpha\#_\sigma H) \mid \alpha \in \Omega\} \tag{9}$$

where I_α is an H-ideal of R for all $\alpha \in \Omega$;
(7) $\Phi(r_b(R)) = r_b(A\#H^*)$;
(8) $\Phi(r_j(R)) = r_j(A\#H^*)$;
(9) $\Phi(r_{Hb}(R)) = r_{Hb}(A\#H^*) = (r_{Hb}(R)\#_\sigma H)\#H^*$.

Proof (1) By [10, Corollary 8.3.11], we have that $P = (P \cap R)A = (P \cap R)\#_\sigma H$.
(2) By the part (b) in the proof of [9, Theorem 7.2], it follows that

$$\Phi(I) = (I\#_\sigma H)\#H^*$$

for every H-ideal I of R.

(3) Obviously, the left side of relation (5) contains the right side of relation (5). If P is an H-ideal of $A\#H^*$, then $P = (P\cap A)\#H^* = (((P\cap A)\cap R)\#_\sigma H)\#H^*$ by part (1), which implies that the right side of relation (5) contains the left side. Thus relation (5) holds. Similarly, relation (6) holds. Now, we show that relation (7) holds. If P is an H-prime ideal of $A\#H^*$, there exists an H-ideal I of R such that $P = (I\#_\sigma H)\#H^*$ by relation (5). For any H-ideals J and J' of R with $JJ' \subseteq I$, since $\Phi(JJ') = \Phi(J)\Phi(J') \subseteq \Phi(I) = P$ by Lemma 2.1 (1), we have that $\Phi(J) \subseteq \Phi(I)$ or $\Phi(J') \subseteq \Phi(I)$, which implies that $J \subseteq I$ or $J' \subseteq I$ by Lemma 2.1. Thus I is an H-prime ideal of R. Conversely, if I is an H-prime ideal of R and $P = (I\#_\sigma H)\#H^*$, we claim that P is an

H-prime of $A\#H^*$. For any H-ideals Q and Q' of $A\#H^*$ with $QQ' \subseteq P$, there exist two H-ideals J and J' of R such that $(J\#_\sigma H)\#H^* = Q$ and $(J'\#_\sigma H)\#H^* = Q'$ by relation (5). Since $\Phi(JJ') = \Phi(J)\Phi(J') = QQ' \subseteq P = \Phi(I)$, $JJ' \subseteq I$, which implies $J \subseteq I$, or $J' \subseteq I$, and so $Q \subseteq P$ or $Q' \subseteq P$. Thus P is an H-prime ideal of $A\#H^*$. Consequently, relation (7) holds.

(4) It follows from [9, Lemma 7.3 (1) (2)].

(5) Obviously, the right side of relation (8) contains the left side. Conversely, if $x \in \cap\{(I_\alpha : H) \mid \alpha \in \Omega\}$, then $x \in (I_\sigma : H)$ and $h \cdot x \in I_\alpha$ for all $\alpha \in \Omega, h \in H$, which implies that $h \cdot x \in \cap\{I_\alpha \mid \alpha \in \Omega\}$ and $x \in (\cap\{I_\alpha \mid \alpha \in \Omega\} : H)$. Thus relation (8) holds.

(6) Let $\{h^{(1)}, \cdots, h^{(n)}\}$ be a basis of H. Obviously, the right side of relation (9) contains the left side of relation (9). Conversely, for $u \in \cap\{(I_\alpha\#_\sigma H) \mid \alpha \in \Omega\}$ and $\alpha, \alpha' \in \Omega$, there exist $r_i \in I_\alpha$ and $r_i' \in I_{\alpha'}$ such that $u = \sum_{i=1}^n r_i\#h^{(i)} = \sum_{i=1}^n r_i'\#h^{(i)}$. Since $\{h^{(1)}, \cdots, h^{(n)}\}$ is linearly independent, we have that $r_i = r_i'$, which implies that $u \in (\cap\{I_\alpha \mid \alpha \in \Omega\})\#_\alpha H$. Thus relation (9) holds.

(7) and (8) follow from Lemma 2.1(4)(5).

(9) We see that

$$\begin{aligned}
r_{Hb}(A\#H^*) &= \cap\{P \mid P \text{ is an } H\text{-prime ideal of } A\#H^*\} \quad \text{by Theorem 1.4} \\
&= \cap\{(I\#_\sigma H)\#H^* \mid I \text{ is an } H\text{-prime ideal of } R\} \quad \text{by relation (7)} \\
&= (\cap\{I\#_\sigma H \mid I \text{ is an } H\text{-prime ideal of } R\})\#H^* \quad \text{by part (6)} \\
&= ((\cap\{I \mid I \text{ is an } H\text{-prime ideal of } R\})\#_\sigma H)\#H^* \quad \text{by part (6)} \\
&= (r_{Hb}(R)\#_\sigma H)\#H^* \quad \text{by Theorem 1.4} \\
&= \Phi(r_{Hb}(R)) \quad \text{by part (2)} .
\end{aligned}$$

(10) If H is cosemisimple, then H is semisimple by [10, Theorem 2.5.2]. Conversely, if H is semisimple, then H^* is cosemisimple. By [10, Theorem 2.5.2], H^* is semisimple. Thus H is cosemisimple. □

Proposition 2.3 *(1)* $r_{Hb}(R) \subseteq r_b(R\#_\sigma H) \cap R \subseteq r_{bH}(R)$*;*

(2) $r_{Hb}(R)\#_\sigma H \subseteq r_b(R\#_\sigma H)$.

Proof. (1) If P is a prime ideal of $R\#_\sigma H$, then $P \cap R$ is an H-prime ideal of R by [5, Lemma 1.6]. Thus $r_b(R\#_\sigma H) \cap R = \cap\{P \cap R \mid P \text{ is prime ideal of } R\#_\sigma H\} \supseteq r_{Hb}(R)$. For any $a \in r_b(R\#_\sigma H) \cap R$ and any m-sequence $\{a_i\}$ in R with $a_1 = a$, it is easy to check that $\{a_i\}$ is also an m-sequence in $R\#_\sigma H$. Thus $a_n = 0$ for some natural n, which implies $a \in r_b(R)$. Thus $r_b(R\#_\sigma H) \cap R \subseteq r_{bH}(R)$ by [2, Lemma 1.6]

(2) We see that

$$\begin{aligned} r_{Hb}(R)\#_\sigma H &= (r_{Hb}(R)\#_\sigma 1)(1\#_\sigma H) \\ &\subseteq r_b(R\#_\sigma H)(1\#_\sigma H) \quad \text{by part (1)} \\ &\subseteq r_b(R\#_\sigma H). \quad \square \end{aligned}$$

Proposition 2.4 *Let H be finite-dimensional Hopf algebra and $A = R\#_\sigma H$. Then*
(1) $r_{H^*b}(R\#_\sigma H) = r_{Hb}(R)\#_\sigma H$;
(2) $r_{Hb}(R) = r_{bH}(R) = r_b(R\#_\sigma H) \cap R$.

Proof

(1) We see that

$$\begin{aligned} r_{H^*b}(R\#_\sigma H) &= \cap\{P \mid P \text{ is an } H^*\text{-prime ideal of } A\} \\ &= \cap\{I\#_\sigma H \mid I \text{ is an } H\text{-prime ideal of } R\} \quad (\text{ by [9, Lemma 7.3 (4)] }) \\ &= (\cap\{I \mid I \text{ is an } H\text{-prime ideal of } R\})\#_\sigma H \quad (\text{ by Lemma 2.2 (6)}) \\ &= r_{Hb}(R)\#_\sigma H. \end{aligned}$$

(2) We see that

$$\begin{aligned} r_{Hb}(R) &= \cap\{P \mid P \textit{ is an } H\text{-prime ideal of } R\} \\ &= \cap\{(I : H) \mid I \in \textit{Spec}(R)\} \ \text{ by Lemma 2.2 part (4)} \\ &= (\cap\{I \mid I \in \textit{Spec}(R)\} : H) \ \text{ by Lemma 2.2 part (5)} \\ &= (r_b(R) : H) \\ &= r_{bH}(R). \end{aligned}$$

Thus it follows from Proposition 2.3(1) that $r_{Hb}(R) = r_b(R\#_\sigma H) \cap R = r_{bH}(R)$. $\square$

Theorem 2.5 . *Let H be a finite-dimensional Hopf algebra and the weak action of H be inner. Then*
(1) $r_{Hb}(R) = r_b(R) = r_{bH}(R)$;
Moreover, if H is semisimple, then
(2) $r_b(R\#_\sigma H) = r_{Hb}(R)\#_\sigma H$.

Proof (1) Since the weak action is inner, every ideal of R is an H-ideal, which implies that $r_{Hb}(R) = r_b(R) = r_{bH}(R)$ by Proposition 2.4 (2).

(2) Considering Proposition 2.3(2), it suffices to show $r_b(R\#_\sigma H) \subseteq r_{Hb}(R)\#_\sigma H$. It is clear that

$$(R\#_\sigma H)/(r_{Hb}(R)\#_\sigma H) \cong (R/r_{Hb}(R))\#_\sigma H \qquad (\text{ as algebras }). \tag{10}$$

It follows by [10, Theorem 7.4.7] that $(R/r_{Hb}(R))\#_\sigma H$ is semiprime. Therefore

$$r_b(R\#_\sigma H) \subseteq r_{Hb}(R)\#_\sigma H. \ \square$$

Theorem 2.6 *Let H be a finite-dimensional, semisimple and either commutative or cocommutative Hopf algebra and let $A = R\#_\sigma H$. Then*

(1) $r_b(R\#_\sigma H) = r_{Hb}(R)\#_\sigma H$;

(2) R is H semiprime iff $R\#_\sigma H$ is semiprime.

Moreover, if H is cosemisimple, or char k does not divide dim H, then both part (3) and part (4) hold:

(3) $r_{Hb}(R) = r_{bH}(R) = r_b(R)$;

(4) R is H-semiprime iff R is semiprime iff $R\#_\sigma H$ is semiprime.

Proof (1) Considering Proposition 2.3(2), it suffices to show

$$r_b(R\#_\sigma H) \subseteq r_{Hb}(R)\#_\sigma H.$$

It follows by [9, Theorem 7.12 (3)] that $(R/r_{Hb}(R))\#_\sigma H$ is semiprime. Using relation (10), we have that $r_b(R\#_\sigma H) \subseteq r_{Hb}(R)\#_\sigma H$.

(2) It follows from part (1) and Proposition 1.3 (1).

(3) By [8, Theorem 4.3 (1)], we have that H is semisimple and cosemisimple.

We see that

$$\begin{aligned}
\Phi(r_b(R)) &= r_b(A\#H^*) \quad \text{by Lemma 2.2 (7)} \\
&= r_{H^*b}(A)\#H^* \quad \text{by part (1)} \\
&= (r_{Hb}(R)\#_\sigma H)\#H^* \quad \text{by Proposition 2.4 (1)} \\
&= \Phi(r_{Hb}(R)) \quad \text{by Lemma 2.2 (2).}
\end{aligned}$$

Thus $r_b(R) = r_{Hb}(R)$.

(4) It immediately follows from part (2) and part (3). $\square$

We now provide an example to show that the Baer radical $r_b(R)$ of R is not H-stable when H is not cosemisimple.

Example: Let k be a field of characteristic $p > 0$ and $R = k[x]/(x^p)$. Then we can define a derivation d on R by sending x to $x + 1$. Then $d^2(x) = d(x + 1) = d(x)$ and

then, by induction, $d^p(x) = d(x)$. It follows that $d^p = d$ on all of R.Thus $H = u(kd)$, the restricted enveloping algebra, is semisimple by [10, Theorem 2.3.3]. clearly H acts on R, but H does not stabilize the Baer radical of R which is the principal ideal generated by x. Note also that H is commutative and cocommutative.

Proposition 2.7 *If R is an H-module algebra, then*

$$r_{Hj}(R) = \cap\{(0 : M)_R \mid M \textit{ is an irreducible } R\text{-}H\textit{-module}\}.$$

That is, $r_{Hj}(R)$ is the H-Jacobson radical of the H-module algebra R defined in [7].

Proof. It is easy to show that M is an irreducible R-H-module iff M is an irreducible $R\#H$-module by [7, Lemma 1]. Thus

$$\begin{aligned} r_{Hj}(R) &= r_j(R\#H) \cap R \quad \text{by definition 1.1} \\ &= (\cap\{(0 : M)_{R\#H} \mid M \text{ is an irreducible } R\#H\text{-module}\}) \cap R \\ &= \cap\{(0 : M)_R \mid M \text{ is an irreducible } R\text{-}H\text{-module}\}.\square \end{aligned}$$

Proposition 2.8 *(1) $r_j(R\#_\sigma H) \cap R = r_{Hj}(R) \subseteq r_{jH}(R)$;*
(2) $r_{Hj}(R)\#_\sigma H \subseteq r_j(R\#_\sigma H)$.

Proof. (1) For any $a \in r_j(R\#_\sigma H) \cap R$, there exists $u = \sum_i a_i\#h_i \in R\#_\sigma H$ such that

$$a + u + au = 0.$$

Let $(id \otimes \epsilon)$ act on the above equation. We get that $a + \sum a_i\epsilon(h_i) + a(\sum a_i\epsilon(h_i)) = 0$, which implies that a is a right quasi-regular element in R. Thus $r_j(R\#_\sigma H)\cap R \subseteq r_{jH}(R)$.

(2) It is similar to the proof of Proposition 2.3 (2). $\square$

Proposition 2.9 *Let H be a finite-dimensional Hopf algebra and $A = R\#_\sigma H$. Then*
(1) $r_{jH}(R)\#_\sigma H = r_{H^*j}(R\#_\sigma H)$;
(2) $r_{Hj}(R) = r_{jH}(R)$;
(3) $r_{Hj}(A\#H^*) = (r_{Hj}(R)\#_\sigma H)\#H^*$.

Proof (1) We see that

$$\begin{aligned} (r_{jH}(R)\#_\sigma H)\#H^* &= \Phi(r_{jH}(R)) \\ &= (\Phi(r_j(R)) \cap A)\#H^* \quad \text{by [9, Theorem 7.2]} \\ &= (r_j(A\#H^*) \cap A)\#H^* \quad \text{by Lemma 2.2 (8)} \\ &= r_{H^*j}(A)\#H^* \quad \text{by Definition 1.1 .} \end{aligned}$$

Thus $r_{H^*j}(A) = r_{jH}(R)\#_\sigma H$.

(2) We see that

$$\begin{aligned} r_{Hj}(R) &= r_j(A) \cap R \\ &\supseteq r_{H^*j}(A) \cap R \ \text{ by Proposition 2.8 (1)} \\ &= r_{jH}(R) \ \text{ by part (1)} . \end{aligned}$$

It follows by Proposition 2.1 (1) that $r_{Hj}(R) = r_{jH}(R)$.

(3) It immediately follows from part (1) (2). $\Box$

By Proposition 2.8 and 2.9, it is clear that if H is a finite-dimensional Hopf algebra, then relation (2) holds iff relation (3) holds.

Theorem 2.10 *Let H be a finite-dimensional Hopf algebra and the weak action of H be inner. Then*

(1) $r_{Hj}(R) = r_j(R) = r_{jH}(R)$.

Moreover, if H is semisimple, then

(2) $r_j(R\#_\sigma H) = r_{Hj}(R)\#_\sigma H$.

Proof (1) Since the weak action is inner, every ideal of R is an H-ideal and $r_j(R) = r_{jH}(R)$. It follows from Proposition 2.2(2) that $r_{Hj}(R) = r_{jH}(R) = r_j(R)$.

(2) Considering Proposition 2.8(2), it suffices to show

$$r_j(R\#_\sigma H) \subseteq r_{Hj}(R)\#_\sigma H.$$

It is clear that

$$(R\#_\sigma H)/(r_{Hj}(R)\#_\sigma H) \cong (R/r_{Hj}(R))\#_\sigma H \qquad \text{(as algebras)}.$$

It follows by [10, Corollary 7.4.3] and part (1) that $(R/r_{Hj}(R))\#_\sigma H$ is semiprimitive. Therefore $r_j(R\#_\sigma H) \subseteq r_{Hj}(R)\#_\sigma H$. $\Box$

Theorem 2.11 *Let H be a finite-dimensional, semisimple Hopf algebra, let k be an algebraically closed field and let $A = R\#_\sigma H$. Assume H is cosemisimple or char k does not divide dim H.*

(1) If H is cocommutative, then

$$r_{Hj}(R) = r_{jH}(R) = r_j(R);$$

(2) If H is commutative, then

$$r_j(R\#_\sigma H) = r_{Hj}(R)\#_\sigma H.$$

Proof By [8, Theorem 4.3 (1)] , we have that H is semisimple and cosemisimple.

(1) If $g \in G(H)$, then the weak action of g on R is an algebraic homomorphism, which implies that $g \cdot r_j(R) \subseteq r_j(R)$. Let H_0 be the coradical of $H, H_1 = H_0 \wedge H_0, H_{i+1} = H_0 \wedge H_i$ for $i = 1, \cdots, n$, where n is the dimension $dimH$ of H. It is clear that $H_0 = kG$ with $G = G(H)$ by [14, Theorem 8.0.1 (c)] and $H = \cup H_i$. It is easy to show that if $k > i$, then

$$H_i \cdot (r_j(R))^k \subseteq r_j(R)$$

by induction for i. Thus

$$H \cdot (r_j(R))^{dimH+1} \subseteq r_j(R),$$

which implies that $(r_j(R))^{dimH+1} \subseteq r_{jH}(R)$.

We see that

$$\begin{aligned} r_j(R/r_{jH}(R)) &= r_j(R)/r_{jH}(R) \\ &= r_b(R/r_{jH}(R)) \quad \text{since} \quad r_j(R)/r_{jH}(R) \quad \text{is} \quad \text{nilpotent} \\ &= r_{bH}(R/r_{jH}(R)) \quad \text{by} \quad \text{Theorem} \quad 2.6\ (3) \\ &\subseteq r_{jH}(R/r_{jH}(R)) \\ &= 0 \quad . \end{aligned}$$

Thus $r_j(R) \subseteq r_{jH}(R)$, which implies that $r_j(R) = r_{jH}(R)$.

(2) It immediately follows from part (1) and Proposition 2.9(1) (2). □

3 The general theory of H-radicals for twisted H-module algebras

In this section we give the general theory of H-radicals for twisted H-module algebras.

Definition 3.1 *Let r be a property of H-ideals of twisted H-module algebras. An H-ideal I of twisted H-module algebra R is called an r-H-ideal of R if it is of the r-property. A twisted H-module algebra R is called an r-twisted H-module algebra if it is r-H-ideal of itself. A property r of H-ideals of twisted H-module algebras is called an H-radical property if the following conditions are satisfied:*

(R1) Every twisted H-homomorphic image of r-twisted H-module algebra is an r twisted H-module algebra;

(R2) Every twisted H-module algebra R has the maximal r-H-ideal $r(R)$;

(R3) $R/r(R)$ has not any non-zero r-H-ideal.

We call $r(R)$ the H-radical of R.

Proposition 3.2 *Let r be an ordinary hereditary radical property for rings. An H-ideal I of twisted H-module algebra R is called an r_H-H-ideal of R if I is an r-ideal of ring R. Then r_H is an H-radical property for twisted H-module algebras.*

Proof. (R1). If (R, σ) is an r_H-twisted H-module algebra and $(R, \sigma) \stackrel{f}{\sim} (R', \sigma')$, then $r(R') = R'$ by ring theory. Consequently, R' is an r_H-twisted H-module algebra.

(R2). For any twisted H-module algebra R, $r(R)$ is the maximal r-ideal of R by ring theory. It is clear that $r(R)_H$ is the maximal r-H-ideal, which is an r_H-H-ideal of R. Consequently, $r_H(R) = r(R)_H$ is the maximal r_H-H-ideal of R.

(R3). If $I/r_H(R)$ is an r_H-H-ideal of $R/r_H(R)$, then I is an r-ideal of algebra R by ring theory. Consequently, $I \subseteq r(R)$ and $I \subseteq r_H(R)$. □

Proposition 3.3 *r_{Hb} is an H-radical property.*

Proof. (R1). Let (R, σ) be an r_{Hb}-twisted H-module algebra and $(R, \sigma) \stackrel{f}{\sim} (R'\sigma')$. For any $x' \in R'$ and any H-m-sequence $\{a'_n\}$ in R' with $a'_1 = x'$, there exist $b'_n \in R'$ and $h_n, h'_n \in H$ such that $a'_{n+1} = (h_n \cdot a'_n)b'_n(h'_n \cdot a'_n)$ for any natural number n. Let $a_1, b_i \in R$ such that $f(a_1) = x'$ and $f(b_i) = b'_i$ for $i = 1, 2, \cdots$. Set $a_{n+1} = (h_n \cdot a_n)b_n(h'_n \cdot a_n)$ for any natural number n. Since $\{a_n\}$ is an H-m-sequence in R, there exists a natural number k such that $a_k = 0$. It is clear that $f(a_n) = a'_n$ for any natural number n by induction. Thus $a'_k = 0$, which implies that x' is an H-m-nilpotent element. Consequently, R' is an r_{Hb}-twisted H-module algebra.

(R2). By [17, Theorem 1.5], $r_{Hb}(R) = W_H(R) = \{a \mid a$ is an H-m-nilpotent element in $R\}$. Thus $r_{Hb}(R)$ is the maximal r_{Hb}-H-ideal of R.

(R3). It immediately follows from [17, Proposition 1.4]. □

4 The relations among radical of R , radical of $R\#_\sigma H$, and H-radical of R

In this section we give the relation among the Jacobson radical $r_j(R)$ of R ,the Jacobson radical $r_j(R\#_\sigma H)$ of $R\#_\sigma H$, and H-Jacobson radical $r_{Hj}(R)$ of R.

In this section, let k be a field, R an algebra with unit, H a Hopf algebra over k and $R\#_\sigma H$ an algebra with unit. Let r be a hereditary radical property for rings which satisfies

$$r(M_{n\times n}(R)) = M_{n\times n}(r(R))$$

for any twisted H-module algebra R.

Example. r_j, r_{bm} and r_n satisfy the above conditions by [15]. Using [17, Lemma 2.1 (2)],we can easily prove that r_b and r_l also satisfy the above conditions.

Definition 4.1 *$\bar{r}_H(R) := r(R\#_\sigma H) \cap R$ and $r_H(R) := (r(R) : H)$.*

If H is a finite-dimensional Hopf algebra and $M = R\#_\sigma H$, then M is a free right R-module with finite rank by [10, Proposition 7.2.11] and $End(M_R) \cong (R\#_\sigma H)\#H^*$ by [10, Corollary 9.4.17]. It follows from part (a) in the proof of [9, Theorem 7.2] that there exists a unique bijective map

$$\Phi : \mathcal{I}(R) \longrightarrow \mathcal{I}(R')$$

such that $\Phi(I)M = MI$, where $R' = (R\#_\sigma H)\#H^*$ and $\mathcal{I}(R) = \{I \mid I \text{ is an ideal of } R\}$.

Lemma 4.2 *If H is a finite-dimensional Hopf algebra, then $\Phi(r(R)) = r((R\#_\sigma H)\#H^*)$.*

Proof. It is similar to the proof of [17, lemma 2.1 (5)]. □

Proposition 4.3 *$\bar{r}_H(R)\#_\sigma H \subseteq r_{H^*}(R\#_\sigma H) \subseteq r(R\#_\sigma H)$.*

Proof. We see that

$$\begin{aligned}
\bar{r}_H(R)\#_\sigma H &= (\bar{r}_H(R)\#_\sigma 1)(1\#_\sigma H) \\
&\subseteq r(R\#_\sigma H)(1\#_\sigma H) \\
&\subseteq r(R\#_\sigma H).
\end{aligned}$$

Thus $\bar{r}_H(R)\#_\sigma H \subseteq r_{H^*}(R\#_\sigma H)$ since $\bar{r}_H(R)\#_\sigma H$ is an H^*-ideal of $R\#_\sigma H$. □

Proposition 4.4 *If H is a finite-dimensional Hopf algebra, then*
(1) $r_H(R)\#_\sigma H = \bar{r}_{H^*}(R\#_\sigma H)$;
Furthermore, if $\bar{r}_H \leq r_H$, then
(2) $\bar{r}_H = r_H$ and $r_H(R)\#_\sigma H \subseteq r(R\#_\sigma H)$;
(3) $R\#_\sigma H$ is r-semisimple for any r_H-semisimple R iff

$$r(R\#_\sigma H) = r_H(R)\#_\sigma H.$$

Proof. Let $A = R\#_\sigma H$.

(1) We see that

$$\begin{aligned}
(r_H(R)\#_\sigma H)\#H^* &= \Phi(r_H(R)) \\
&= (\Phi(r(R)) \cap A)\#H^* \quad \text{by [9, Theorem 7.2]} \\
&= (r(A\#H^*) \cap A)\#H^* \quad \text{by Lemma 4.2} \\
&= \bar{r}_{H^*}(A)\#H^* \quad \text{by Definition 4.1 .}
\end{aligned}$$

Thus $\bar{r}_{H^*}(A) = r_H(R)\#_\sigma H$.

(2) We see that

$$\begin{aligned} \bar{r}_H(R) &= r(A) \cap R \\ &\supseteq \bar{r}_{H^*}(A) \cap R \ \text{ by assumption} \\ &= r_H(R) \ \text{ by } \ \text{part } \ (1) \ . \end{aligned}$$

Thus $\bar{r}_H(R) = r_H(R)$ by assumption.

(3) Sufficiency is obvious. Now we show the necessity. Since

$$r((R\#_\sigma H)/(r_H(R)\#_\sigma H)) \cong r(R/r_H(R)\#_{\sigma'} H) = 0,$$

we have $r(R\#_\sigma H) \subseteq r_H(R)\#_\sigma H$. Considering part (2), we have

$$r(R\#_\sigma H) = r_H(R)\#_\sigma H. \quad \square$$

Corollary 4.5 *Let r denote r_b, r_l, r_j, r_{bm} and r_n. Then*

(1) $\bar{r}_H \leq r_H$;

Furthermore, if H is a finite-dimensional Hopf algebra, then

(2) $\bar{r}_H = r_H$;

(3) $R\#_\sigma H$ is r- semisimple for any r_H-semisimple R iff $r(R\#_\sigma H) = r_H(R)\#_\sigma H$;

(4) $R\#_\sigma H$ is r_j- semisimple for any r_{Hj}-semisimple R iff $r_j(R\#_\sigma H) = r_{Hj}(R)\#_\sigma H$.

Proof. (1) When $r = r_b$ or $r = r_j$, it has been proved in [17, Proposition 2.3 (1) and 3.2 (1)] and in the preceding sections. The others can similarly be proved.

(2) It follows from Proposition 4.4 (2).

(3) and (4) follow from part (1) and Proposition 4.4 (3). $\square$

Proposition 4.6 *If $H = kG$ or the weak action of H on R is inner, then*

(1). $r_H(R) = r(R)$;

(2) If, in addition, H is a finite-dimensional Hopf algebra and $\bar{r}_H \leq r_H$, then $r_H(R) = \bar{r}_H(R) = r(R)$.

Proof. (1) It is trivial.

(2) It immediately follows from part (1) and Proposition 4.1 (1) (2). $\square$

Theorem 4.7 *Let G be a finite group and $\mid G \mid^{-1} \in k$. If $H = kG$ or $H = (kG)^*$, then*

(1) $r_j(R) = r_{Hj}(R) = r_{jH}(R)$;

(2) $r_j(R\#_\sigma H) = r_{Hj}(R)\#_\sigma H$.

Proof. (1) Let $H = kG$. We can easily check $r_j(R) = r_{jH}(R)$ using the method similar to the proof of [16, Proposition 4.6]. By [17, Proposition 3.3 (2)], $r_{Hj}(R) = r_{jH}(R)$. Now, we only need to show that

$$r_j(R) = r_{H^*j}(R).$$

We see that

$$\begin{aligned} r_j((R\#_\sigma H^*)\#H) &= r_{H^*j}((R\#_\sigma H^*)\#H) \quad \text{by [6, Theorem 4.4 (3)]} \\ &= r_{Hj}(R\#_\sigma H^*)\#H \quad \text{by [17, Proposition 3.3 (1)]} \\ &= (r_{H^*j}(R)\#_\sigma H^*)\#H \quad \text{by [17, Proposition 3.3 (1)].} \end{aligned}$$

On the one hand, by [17, Lemma 2.2 (8)], $\Phi(r_j(R)) = r_j((R\#_\sigma H^*)\#H)$. On the other hand, we have that $\Phi(r_{H^*j}(R)) = (r_{H^*j}(R)\#_\sigma H^*)\#H$ by [17, Lemma 2.2 (2)]. Consequently, $r_j(R) = r_{H^*j}(R)$.

(2) It immediately follows from part (1) and [17, Proposition 3.3 (1) (2)]. □

Corollary 4.8 *Let H be a semisimple and cosemisimple Hopf algebra over algebraically closed field k. If H is commutative or cocommutative, then*

$$r_j(R) = r_{Hj}(R) = r_{jH}(R) \qquad \textit{and} \qquad r_j(R\#_\sigma H) = r_{Hj}(R)\#_\sigma H.$$

Proof. It immediately follows from Theorem 4.7 and [14, Lemma 8.0.1 (c)]. □

We give an example to show that conditions in Corollary 4.8 can not be omitted.

Example 4.9 *(see [17, Example P20]) Let k be a field of characteristic $p > 0$, $R = k[x]/(x^p)$. We can define a derivation on R by sending x to $x + 1$. Set $H = u(kd)$, the restricted enveloping algebra, and $A = R\#H$. Then*

(1) $r_b(A\#H^*) \neq r_{H^*b}(A)\#H^*$;

(2) $r_j(A\#H^*) \neq r_{H^*j}(A)\#H^*$;

(3) $r_j(A\#H^*) \not\subseteq r_{jH^*}(A)\#H^*$.

Proof. (1) By [17, Example P20], we have $r_b(R) \neq 0$ and $r_{bH}(R) = 0$. Since $\Phi(r_b(R)) = r_b(A\#H^*) \neq 0$ and $\Phi(r_{bH}(R)) = r_{bH^*}(A)\#H^* = 0$, we have that part (1) holds.

(3) We see that $r_j(A\#H^*) = \Phi(r_j(R))$ and $r_{Hj}(A)\#H^* = \Phi(r_{Hj}(R))$. Since R is commutative, $r_j(R) = r_b(R)$. Thus $r_{Hj}(R) = r_{jH}(R) = r_{bH}(R) = 0$ and $r_j(R) = r_b(R) \neq 0$, which implies $r_j(A\#H^*) \not\subseteq r_{jH^*}(A)\#H^*$.

(2) It follows from part (3). □

This example also answer the question J.R. Fisher asked in [7] :

$$\text{Is} \quad r_j(R\#H) \subseteq r_{jH}(R)\#H \quad ?$$

If F is an extension field of k, we write R^F for $R \otimes_k F$ (see [9, P49]) .

Lemma 4.10 *If F is an extension field of k, then*

(1) H is a semisimple Hopf algebra over k iff H^F is a semisimple Hopf algebra over F;

(2) Furthermore, if H is a finite-dimensional Hopf algebra, then H is a cosemisimple Hopf algebra over k iff H^F is a cosemisimple Hopf algebra over F.

Proof. (1) It is clear that $\int_H^l \otimes F = \int_{H^F}^l$. Thus H is a semisimple Hopf algebra over k iff H^F is a semisimple Hopf algebra over F.

(2) $(H \otimes F)^* = H^* \otimes F$ since $H^* \otimes F \subseteq (H \otimes F)^*$ and $dim_F(H \otimes F) = dim_F(H^* \otimes F) = dim_k H$. Thus we can obtain part (2) by Part (1). $\square$

By the way, if H is a semisimple Hopf algebra, then H is a separable algebra by Lemma 4.10 (see [12, P284]).

Proposition 4.11 *Let F be an algebraic closure of k, R an algebra over k and*

$$r(R \otimes_k F) = r(R) \otimes_k F.$$

If H is a finite-dimensional Hopf algebra with cocommutative coradical over k , then

$$r(R)^{dimH} \subseteq r_H(R).$$

Proof. It is clear that H^F is a finite-dimensional Hopf algebra over F and $dimH = dimH^F = n$. Let H_0^F be the coradical of H^F, $H_1^F = H_0^F \wedge H_0^F, H_{i+1}^F = H_0^F \wedge H_i^F$ for $i = 1, \cdots, n-1$. Notice $H_0^F \subseteq H_0 \otimes F$. Thus H_0^F is cocommutative. It is clear that $H_0^F = kG$ by [14, Lemma 8.0.1 (c)] and $H^F = \cup H_i^F$. It is easy to show that if $k > i$, then

$$H_i^F \cdot (r(R^F))^k \subseteq r(R^F)$$

by induction for i. Thus

$$H^F \cdot (r(R^F))^{dimH} \subseteq r(R^F),$$

which implies that $(r(R^F))^{dimH} \subseteq r(R^F)_{H^F}$. By assumption, we have that $(r(R) \otimes F)^{dimH} \subseteq (r(R) \otimes F)_{H^F}$. It is clear that $(I \otimes F)_{H^F} = I_H \otimes F$ for any ideal I of R. Consequently, $(r(R))^{dimH} \subseteq r(R)_H$. $\square$

Theorem 4.12 *Let H be a semisimple, cosemisimple and either commutative or cocommutative Hopf algebra over k. If there exists an algebraic closure F of k such that*

$$r_j(R \otimes F) = r_j(R) \otimes F \quad \text{and} \quad r_j((R\#_\sigma H) \otimes F) = r_j(R\#_\sigma H) \otimes F,$$

then

(1) $r_j(R) = r_{Hj}(R) = r_{jH}(R)$;

(2) $r_j(R\#_\sigma H) = r_{Hj}(R)\#_\sigma H$.

Proof. (1). By Lemma 4.10, H^F is semisimple and cosemisimple. Considering Corollary 4.8, we have that $r_j(R^F) = r_{H^F j}(R^F) = r_{jH^F}(R^F)$. On the one hand, by assumption, $r_j(R^F) = r_j(R) \otimes F$. On the other hand, $r_{jH^F}(R^F) = (r_j(R) \otimes F)_{H^F} = r_{jH}(R) \otimes F$. Thus $r_j(R) = r_{jH}(R)$.

(2). It immediately follows from part (1). $\Box$

Considering Theorem 4.12 and [12, Theorem 7.2.13], we have

Corollary 4.13 *Let H be a semisimple, cosemisimple and either commutative or cocommutative Hopf algebra over k. If there exists an algebraic closure F of k such that F/k is separable and algebraic, then*

(1) $r_j(R) = r_{Hj}(R) = r_{jH}(R)$;

(2) $r_j(R\#_\sigma H) = r_{Hj}(R)\#_\sigma H$.

Lemma 4.14 *(Szasz [15])*

$$r_j(R) = r_k(R)$$

holds in the following three cases:

(1) Every element in R is algebraic over k ([15, Proposition 31.2]);

(2) The cardinality of k is strictly greater than the dimension of R and k is infinite ([15, Theorem 31.4]);

(3) k is uncountable and R is finitely generated ([15, Proposition 31.5]).

Proposition 4.15 *Let F be an extension of k. Then $r(R) \otimes F \subseteq r(R \otimes F)$, where r denotes r_b, r_k, r_l, r_n.*

Proof. When $r = r_n$, for any $x \otimes a \in r_n(R) \otimes F$ with $a \neq 0$, there exists $y \in R$ such that $x = xyx$. Thus $x \otimes a = (x \otimes a)(y \otimes a^{-1})(x \otimes a)$, which implies $r_n(R) \otimes F \subseteq r_n(R \otimes F)$.

Similarly, we can obtain the others. $\Box$

Corollary 4.16 *Let H be a semisimple, cosemisimple and commutative or cocommutative Hopf algebra. If there exists an algebraic closure F of k such that F/k is a pure transcendental extension and one of the following three conditions holds:*

(i) every element in $R\#_\sigma H$ is algebraic over k;

(ii) the cardinality of k is strictly greater than the dimension of R and k is infinite;

(iii) k is uncountable and R is finitely generated;

then

(1) $r_j(R) = r_{Hj}(R) = r_{jH}(R)$;

(2) $r_j(R\#_\sigma H) = r_{Hj}(R)\#_\sigma H$;

(3) $r_j(R) = r_k(R)$ and $r_j(R\#_\sigma H) = r_k(R\#_\sigma H)$.

Proof. First, we have that part (3) holds by Lemma 4.14. We next see that

$$\begin{aligned} r_j(R \otimes F) &\subseteq r_j(R) \otimes F && [12, \text{Theorem 7.3.4}] \\ &= r_k(R) \otimes F && \text{part (3)} \\ &\subseteq r_k(R \otimes F) && \text{proposition 4.15} \\ &\subseteq r_j(R \otimes F). \end{aligned}$$

Thus $r_j(R \otimes F) = r_j(R) \otimes F$. Similarly, we can show that $r_j((R \#_\sigma H) \otimes F) = r_j(R \#_\sigma H) \otimes F$.

Finally, using Theorem 4.12, we complete the proof. □

5 The H-Von Neumann regular radical

In this section, we construct the H-von Neumann regular radical for H-module algebras and show that it is an H-radical property.

Definition 5.1 *Let $a \in R$. If $a \in (H \cdot a)R(H \cdot a)$, then a is called an H-von Neumann regular element, or an H-regular element in short. If every element of R is an H-regular, then R is called an H-regular module algebra, written as r_{Hn}-H-module algebra. I is an H-ideal of R and every element in I is H-regular, then I is called an H- regular ideal.*

Lemma 5.2 *If I is an H-ideal of R and $a \in I$, then a is H-regular in I iff a is H-regular in H.*

Proof. The necessity is clear.

Sufficiency: If $a \in (H \cdot a)R(H \cdot a)$, then there exist $h_i, h_i' \in H, b_i \in R$, such that

$$a = \sum (h_i \cdot a) b_i (h_i' \cdot a).$$

We see that

$$\begin{aligned} a &= \sum_{i,j} [h_i \cdot ((h_j \cdot a) b_j (h_j' \cdot a))] b_i (h_i' \cdot a) \\ &= \sum_{i,j} [((h_i)_1 \cdot (h_j \cdot a))((h_i)_2 \cdot b_j)((h_i)_3 \cdot (h_j' \cdot a))] b_i (h_i' \cdot a) \\ &\in (H \cdot a) I (H \cdot a). \end{aligned}$$

Thus a is an H-regular in I. □

Lemma 5.3 *If $x - \sum_i (h_i \cdot x) b_i (h_i' \cdot x)$ is H-regular, then x is H-regular, where $x, b_i \in R, h_i, h_i' \in H$.*

Proof. Since $x - \sum_i (h_i \cdot x) b_i (h'_i \cdot x)$ is H-regular, there exist $g_i, g'_i \in H, c_i \in R$ such that

$$x - \sum_i (h_i \cdot x) b_i (h'_i \cdot x) = \sum_j (g_j \cdot (x - \sum_i (h_i \cdot x) b_i (h'_i \cdot x))) c_j (g'_j \cdot (x - \sum_i (h_i \cdot x) b_i (h'_i \cdot x))).$$

Consequently, $x \in (H \cdot x) R (H \cdot x)$. □

Definition 5.4

$$r_{Hn}(R) := \{a \in R \mid \quad \textit{the} \;\; H\textit{-ideal } (a) \textit{ generated by } a \textit{ is } H\textit{-regular} \}.$$

Theorem 5.5 $r_{Hn}(R)$ *is an H-ideal of R.*

Proof. We first show that $Rr_{Hn}(R) \subseteq r_{Hn}(R)$. For any $a \in r_{Hn}(R), x \in R$, we have that (xa) is H-regular since $(xa) \subseteq (a)$. We next show that $a - b \in r_{Hn}(R)$ for any $a, b \in r_{Hn}(R)$. For any $x \in (a - b)$, since $(a - b) \subseteq (a) + (b)$, we have that $x = u - v$ and $u \in (a), v \in (b)$. Say $u = \sum_i (h_i \cdot u) c_i (h'_i \cdot u)$ and $h_i, h'_i \in H, c_i \in R$. We see that

$$\begin{aligned} x \quad & - \sum_i (h_i \cdot x) c_i (h'_i \cdot x) \\ = \quad & (u - v) - \sum_i (h_i \cdot (u - v)) c_i (h'_i \cdot (u - v)) \\ = \quad & -v - \sum_i [-(h_i \cdot u) c_i (h'_i \cdot v) - (h_i \cdot v) c_i (h'_i \cdot u) + (h_i \cdot v) c_i (h'_i \cdot v)] \\ \in \quad & (v). \end{aligned}$$

Thus $x - \sum_i (h_i \cdot x) c_i (h'_i \cdot x)$ is H-regular and x is H-regular by Lemma 5.3. Therefore $a - b \in r_{Hn}(R)$. Obviously, $r_{Hn}(R)$ is H-stable. Consequently, $r_{Hn}(R)$ is an H-ideal of R. □

Theorem 5.6 $r_{Hn}(R / r_{Hn}(R)) = 0.$

Proof. Let $\bar{R} = r / r_{Hn}(R)$ and $\bar{b} = b + r_{Hn}(R) \in r_{Hn}(R / r_{Hn}(R))$. It is sufficient to show that $b \in r_{Hn}(R)$. For any $a \in (b)$, it is clear that $\bar{a} \in (\bar{b})$. Thus there exist $h_i, h'_i \in H, \bar{c}_i \in \bar{R}$ such that

$$\bar{a} = \sum_i (h_i \cdot \bar{a}) \bar{c}_i (h'_i \cdot \bar{a}) = \sum_i \overline{(h_i \cdot a) c_i (h'_i \cdot a)}.$$

Thus $a - \sum_i (h_i \cdot a) c_i (h'_i \cdot a) \in r_{Hn}(R)$, which implies that a is H-regular. Consequently, $b \in r_{Hn}(R)$. Namely, $\bar{b} = 0$ and $r_{Hn}(R) = 0$. □

Corollary 5.7 r_{Hn} *is an H-radical property for H-module algebras and $r_{nH} \leq r_{Hn}$.*

Proof. (R1) If $R \overset{f}{\sim} R'$ and R is an r_{Hn} -H-module algebra, then for any $f(a) \in R'$, $f(a) \in (H \cdot f(a))R'(H \cdot f(a))$. Thus R' is also an r_{Hn}- H-module algebra.

(R2) If I is an r_{Hn}-H-ideal of R and $r_{Hn}(R) \subseteq I$ then, for any $a \in I$, (a) is H-regular since $(a) \subseteq I$. Thus $I \subseteq r_{Hn}(R)$.

(R3) It follows from Theorem 5.8.

Consequently r_{Hn} is an H-radical property for H-module algebras. It is straightforward to check $r_{nH} \leq r_{Hn}$. $\square$

r_{Hn} is called the H-von Neumann regular radical.

Theorem 5.8 *If I is an H-ideal of R, then $r_{Hn}(I) = r_{Hn}(R) \cap I$. Namely, r_{Hn} is a strongly hereditary H-radical property.*

Proof. By Lemma 5.2, $r_{Hn}(R) \cap I \subseteq r_{Hn}(I)$. Now, it is sufficient to show that $(x)_I = (x)_R$ for any $x \in r_{Hn}(I)$, where $(x)_I$ and $(x)_R$ denote the H-ideals generated by x in I and R respectively. Let $x = \sum(h_i \cdot x)b_i(h'_i \cdot x)$, where $h_i, h'_i \in H, b_i \in I$. We see that

$$\begin{aligned} R(H \cdot x) &= R(H \cdot (\sum(h_i \cdot x)b_i(h'_i \cdot x)) \\ &\subseteq R(H \cdot x)I(H \cdot x) \\ &\subseteq I(H \cdot x). \end{aligned}$$

Similarly,

$$(H \cdot x)R \subseteq (H \cdot x)I.$$

Thus $(x)_I = (x)_R$. $\square$

A graded algebra R of type G is said to be Gr-regular if for every homogeneous $a \in R_g$ there exists $b \in R$ such that $a = aba$ (see [11] P258). Now, we give the relations between Gr-regularity and H-regularity.

Theorem 5.9 *If G is a finite group, R is a graded algebra of type G, and $H = (kG)^*$, then R is Gr-regular iff R is H-regular.*

Proof. Let $\{p_g \mid g \in G\}$ be the dual base of base $\{g \mid g \in G\}$. If R is Gr-regular for any $a \in R$, then $a = \sum_{g \in G} a_g$ with $a_g \in R_g$. Since R is Gr-regular, there exist $b_{g^{-1}} \in R_{g^{-1}}$ such that $a_g = a_g b_{g^{-1}} a_g$ and

$$a = \sum_{g \in G} a_g = \sum_{g \in G} a_g b_{g^{-1}} a_g = \sum_{g \in G} (p_g \cdot a) b_{g^{-1}} (p_g \cdot a).$$

Consequently, R is H-regular.

Conversely, if R is H-regular, then for any $a \in R_g$, there exists $b_{x,y} \in R$ such that

$$a = \sum_{x,y \in G} (p_x \cdot a) c_{x,y} (p_y \cdot a).$$

Considering $a \in R_g$, we have that $a = ab_{g,g}a$. Thus R is Gr-regular .$\square$

6 About J.R. Fisher's question

In this section, we answer the question J.R. Fisher asked in [7]. Namely, we give a necessary and sufficient condition for validity of relation (2) .

Throughout this section, let k be a commutative ring with unit, R an H- module algebra and H a Hopf algebra over k.

Theorem 6.1 *Let $\mathcal{K}$ be an ordinary special class of rings and closed with respect to isomorphism. Set $r = r^{\mathcal{K}}$ and $\bar{r}_H(R) = r(R\#H) \cap R$ for any H-module algebra R. Then $\bar{r}_H$ is an H-radical property of H-module algebras. Furthermore, it is an H-special radical.*

Proof. Let $\bar{\mathcal{M}}_R = \{M \mid M$ is an R-prime module and $R/(0 : M)_R \in \mathcal{K}\}$ for any ring R and $\bar{\mathcal{M}} = \cup \bar{\mathcal{M}}_R$. Set $\mathcal{M}_R = \{M \mid M \in \bar{\mathcal{M}}_{R\#H}\}$ for any H- module algebra R and $\mathcal{M} = \cup \mathcal{M}_R$. It is straightforward to check that $\bar{\mathcal{M}}$ satisfies the conditions of [16, Proposition 4.3]. Thus $\mathcal{M}$ is an H-special module by [16, Proposition 4.3]. It is clear that $\mathcal{M}(R) = \bar{\mathcal{M}}(R\#H) \cap R = r(R\#H) \cap R$ for any H-module algebra R. Thus $\bar{r}_H$ is an H-special radical by [16, Theorem 3.1]. □

Using the Theorem 6.1, we have that $\bar{r}_{bH}, \bar{r}_{lH}, \bar{r}_{kH}, \bar{r}_{jH}, \bar{r}_{bmH}$ are all H-special radicals.

Proposition 6.2 *Let $\mathcal{K}$ be a special class of rings and closed with respect to isomorphism. Set $r = r^{\mathcal{K}}$. Then*

(1) $\bar{r}_H(R)\#H \subseteq r(R\#H)$;

(2) $\bar{r}_H(R)\#H = r(R\#H)$ iff there exists an H-ideal I of R such that $r(R\#H) = I\#H$;

(3) R is an $\bar{r}_H$-H-module algebra iff $r(R\#H) = R\#H$;

(4) I is an $\bar{r}_H$-H-ideal of R iff $r(I\#H) = I\#H$;

(5) $r(\bar{r}_H(R)\#H) = \bar{r}_H(R)\#H$;

(6) $r(R\#H) = \bar{r}_H(R)\#H$ iff $r(\bar{r}_H(R)\#H) = r(R\#H)$.

Proof. (1) It is similar to the proof of Proposition 4.3.

(2) It is a straightforward verification.

(3) If R is an $\bar{r}_H$-module algebra, then $R\#H \subseteq r(R\#H)$ by part (1). Thus $R\#H = r(R\#H)$. The sufficiency is obvious.

(4), (5) and (6) immediately follow from part (3) . □

Theorem 6.3 *If R is an algebra over field k with unit and H is a Hopf algebra over field k, then*

(1) $\bar{r}_{jH}(R) = r_{Hj}(R)$ and $r_j(r_{Hj}(R)\#H) = r_{Hj}(R)\#H$;

(2) $r_j(R\#H) = r_{Hj}(R)\#H$ iff $r_j(r_{Hj}(R)\#H) = r_j(R\#H)$ iff $r_j(r_j(R\#H) \cap R\#H) = r_j(R\#H)$;

(3) Furthermore, if H is finite-dimensional, then $r_j(R\#H) = r_{Hj}(R)\#H$ iff $r_j(r_{jH}(R)\#H) = r_j(R\#H)$.

Proof. (1) By [17, Proposition 3.1], we have $\bar{r}_{jH}(R) = r_{Hj}(R)$. Consequently, $r_j(r_{Hj}(R)\#H) = r_{Hj}(R)\#H$ by Proposition 6.2 (5).

(2) It immediately follows from part (1) and Proposition 6.2 (6).

(3) It can easily be proved by part (2) and [17, Proposition 3.3 (2)]. $\Box$

The theorem answers the question J.R. Fisher asked in [7] : When is $r_j(R\#H) = r_{Hj}(R)\#H$?

Proposition 6.4 *If R is an algebra over field k with unit and H is a finite-dimensional Hopf algebra over field k, then*

(1) $\bar{r}_{bH}(R) = r_{Hb}(R) = r_{bH}(R)$ *and* $r_b(r_{Hb}(R)\#H) = r_{Hb}(R)\#H$;

(2) $r_b(R\#H) = r_{Hb}(R)\#H$ *iff* $r_b(r_{Hb}(R)\#H) = r_b(R\#H)$ *iff* $r_b(r_{bH}(R)\#H) = r_b(R\#H)$ *iff* $r_b(r_b(R\#H) \cap R\#H) = r_b(R\#H)$.

Proof. (1) By [17, Proposition 2.4], we have $\bar{r}_{bH}(R) = r_{Hb}(R)$. Thus $r_b(r_{Hb}(R)\#H) = r_{Hb}(R)\#H$ by Proposition 6.2 (5).

(2) It follows from part (1) and Proposition 6.2 (6) . $\Box$

In fact, if H is commutative or cocommutative, then $S^2 = id_H$ by [14, Proposition 4.0.1], and H is semisimple and cosemisimple iff the character $chark$ of k does not divides $dimH$ (see [13, Proposition 2 (c)]). It is clear that if H is a finite-dimensional commutative or cocommutative Hopf algebra and the character $chark$ of k does not divides $dimH$, then H is a finite-dimensional semisimple, cosemisimple, commutative or cocommutative Hopf algebra. Consequently, the conditions in Corollary 4.8, Theorem 4.12, Corollary 4.13 and 4.16 can be simplified

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